Finite-size effects in a population of interacting oscillators

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We consider a large population of globally coupled noisy phase oscillators. In the thermodynamic limit $N\to\infty$ this system exhibits a nonequilibrium phase transition, at which a macroscopic mean field appears. It is shown that for large but finite system size N the system can be described by the noisy Stuart-Landau equation, yielding scaling behavior of statistical characteristics of the macroscopic mean field with N. The predictions of the theory are checked numerically. [S1063-651X(99)03802-7]

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Many nonequilibrium systems can be represented as ensembles of interacting self-sustained oscillators. Such models are very popular in biophysics [1,2], but can also be applied to the description of Josephson junctions [3], laser arrays [4], charge density waves [5], chemical reactions [6], etc. In these situations the interaction can be often considered as distance independent, thus leading naturally to a model with global, or mean-field, coupling. Two types of systems are usually considered: In one case all oscillators are deterministic, but have different natural frequencies; in the other case the oscillators are identical, but are driven with uncorrelated noisy forces. The framework of the statistical treatment of such models, as well the models themselves, has been proposed by Kuramoto [7]. The main effect is the transition to mutual synchronization as the coupling strength exceeds a threshold value. Different aspects of this nonequilibrium phase transition have been studied in [8–11].

In this paper we focus on finite-size properties of the transition to mutual synchronization in a population of noisy globally coupled oscillators. We study the statistics of fluctuations that appear for finite ensemble sizes N. A similar question regarding an ensemble of deterministic oscillators with distributed natural frequency has been addressed by Daido [9]. Our approach differs from that of [9], as we assume the gauge invariance of the problem (there is no preferred value of the phase of the oscillations), while the solution given in [9] breaks this invariance.

We consider a phase model of identical globally coupled noisy oscillators [7,12,13]. Each oscillator is described by the phase θ_n, whose dynamics is given by the equation

$$\frac{d\theta_n}{dt}=\varepsilon\frac{1}{N}\sum_{l=1}^{N}\sin(\theta_l-\theta_n)+\xi_n(t). \tag{1}$$

Here $\varepsilon>0$ is the coupling constant, N is the number of oscillators in the population, and $\xi_n(t)$ is Gaussian δ-correlated driving noise

$$\langle\xi_n(t)\rangle=0,\quad \langle\xi_n(t)\xi_m(t')\rangle=2D\,\delta(t-t')\,\delta_{nm}.$$

It is convenient to introduce the complex mean field M as

$$M=X+iY=\frac{1}{N}\sum_{1}^{N}e^{i\theta_n}$$

and to rewrite Eq. (1) as the mean-field governed dynamics

$$\frac{d\theta_n}{dt}=\varepsilon(-X\sin\theta_n+Y\cos\theta_n)+\xi_n(t). \tag{2}$$

It is well known that if the coupling ε exceeds a critical value, a nontrivial state with a finite macroscopic mean field M appears [7]. This transition can be described analytically in the thermodynamic limit $N\to\infty$. Our aim here is to discuss *finite-size effects* that appear for large but finite ensemble sizes N. We start with writing the equations for the quantities

$$C_k=\frac{1}{N}\sum_{n=1}^{N}e^{ik\theta_n}.$$

Note that the mean field is $M=C_1$. Calculating the time derivative, we get a system of ordinary differential equations

$$\frac{dC_k}{dt}=\frac{ik}{N}\sum_{1}^{N}\xi_n e^{ik\theta_n}+\frac{\varepsilon k}{2}(C_1C_{k-1}-C_{-1}C_{k+1}). \tag{3}$$

In these equations only the first term on the right-hand side contains the noise. We estimate this term when the mean field M is small, i.e., in the disordered state, or near the transition point. In this case the effect of the coupling on the dynamics of the phase θ_n is small compared to the effect of the noise, so we can consider the phases θ_n as uncorrelated and the contributions to the term

$$\frac{ik}{N}\sum_{1}^{N}\xi_n e^{ik\theta_n}$$

as independent. Thus we can apply the law of large numbers and write

$$\frac{ik}{N}\sum\xi_n e^{ik\theta_n}\approx A+\eta_k,$$

where A is the deterministic part and η is a deviation having the variance proportional to N^{-1}. To calculate A, we can use the Furutsu-Novikov formula [14,15] to get

$$\langle \xi_n e^{ik\theta_n}\rangle = D\left\langle \frac{\delta e^{ik\theta_n}}{\delta \xi_n}\right\rangle = Dike^{ik\theta_n}.$$

Finally, we obtain the system

$$\frac{dC_k}{dt} = -k^2 D C_k + \frac{\varepsilon k}{2}(C_1 C_{k-1} - C_{-1}C_{k+1}) + \eta_k. \quad (4)$$

For convenience, we write down the equations for the first three modes:

$$\dot{C}_1 = -DC_1 - \frac{\varepsilon}{2}(C_2 C_1^* - C_1) + \eta_1(t),$$

$$\dot{C}_2 = -4DC_2 - \frac{\varepsilon}{2}(C_3 C_1^* - C_1^2) + \eta_2(t), \quad (5)$$

$$\dot{C}_3 = -9DC_3 - \frac{\varepsilon}{2}(C_4 C_1^* - C_1 C_2) + \eta_3(t).$$

In the thermodynamic limit $N \to \infty$ the noisy terms η_k vanish and we get a deterministic system. In this limit the transition is a Hopf bifurcation (with zero frequency of oscillations) in the system (5) and the normal form equation describing this bifurcation (also called the Stuart-Landau equation) can be obtained using the expansion in the small parameter $\varepsilon/2 - D \ll 1$ [7]. Here we follow this approach taking into account noisy terms as well. One can see from Eqs. (5) that the modes C_k with large k decay fast, while the first mode C_1 becomes unstable at the critical coupling $\varepsilon_c = 2D$ and its dynamics is slow. Thus, near the bifurcation point, we can express the higher modes algebraically through the first mode C_1. It is sufficient to assume that $|C_k| \approx 0$ for $k > 2$ and from the condition $\dot{C}_2 \approx 0$ (the second mode decays fast on the time scale of the instability) we get the relation between C_2 and $C_1 : C_2 \approx (\varepsilon/8D) C_1^2 + (1/4D)\eta_2$. Substituting this into the equation for C_1, we get the standard form of the Hopf-Andronov bifurcation

$$\dot{C}_1 = \left(\frac{\varepsilon}{2} - D\right) C_1 - \frac{\varepsilon^2}{16D}|C_1|^2 C_1 - \frac{\varepsilon}{8D}\eta_2 C_1^* + \eta_1. \quad (6)$$

This equation describes the bifurcation in the presence of both multiplicative and additive noise. As the noise intensity is small for large N, near the bifurcation point the multiplicative noise does not lead to additional instability and thus can be neglected when compared to the additive noise. As a result, we get

$$\dot{M} = \left(\frac{\varepsilon}{2} - D\right) M - \frac{\varepsilon^2}{16D}|M|^2 M + \eta_1. \quad (7)$$

In this equation the precise statistical properties of the noise term η_1 are unknown. We can only hypothesize, using the law of large numbers, that it is Gaussian δ correlated with an amplitude proportional to $N^{-1/2}$. We can justify this by the following argument: Near the transition point $\varepsilon \approx \varepsilon_c$ the deterministic part of the dynamics of the mean field is extremely slow, thus the correlation of the noisy term decays fast on the appropriate time scale. So rewriting Eq. (7) as a system of two real equations, we obtain

$$\dot{X} = \left(\frac{\varepsilon}{2} - D\right) X - \frac{\varepsilon^2}{16D}(X^2 + Y^2) X + \eta_x(t), \quad (8)$$

$$\dot{Y} = \left(\frac{\varepsilon}{2} - D\right) Y - \frac{\varepsilon^2}{16D}(X^2 + Y^2) Y + \eta_y(t), \quad (9)$$

with

$$\langle \eta_x \rangle = \langle \eta_y \rangle = 0,$$

$$\langle \eta_x(t)\eta_x(t')\rangle = \langle \eta_y(t)\eta_y(t')\rangle = 2d\,\delta(t-t'),$$

$$\langle \eta_x(t)\eta_y(t')\rangle = 0.$$

The noise intensity scales as $d \propto N^{-1}$, so we can write $d = \sigma N^{-1}$, where $\sigma = O(1)$.

Different statistical characteristics can be determined for the model (8) and (9). Writing out the Fokker-Planck equation (see, e.g., [16]) for the system (8) and (9) in the form

$$\frac{\partial W(X,Y,t)}{\partial t} = \frac{\partial}{\partial X}\left[\frac{\partial V(X,Y)}{\partial X} W\right] + \frac{\partial}{\partial Y}\left[\frac{\partial V(X,Y)}{\partial Y} W\right] + d\left(\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2}\right),$$

with the potential function

$$V(X,Y) = -\left(\frac{\varepsilon}{4} - \frac{D}{2}\right)(X^2 + Y^2) + \frac{\varepsilon^2}{64D}(X^2 + Y^2)^2,$$

one easily gets the stationary solution

$$W_0(X,Y) = c\,\exp\left(-\frac{V(X,Y)}{d}\right) = c\,\exp\left(-\frac{NV(X,Y)}{\sigma}\right). \quad (10)$$

It is convenient to describe statistical properties of the complex mean field M using its representation through the phase Φ and the amplitude R: $M = Re^{i\Phi}$. The stationary distribution (10) is phase independent and after simple calculations we get a scaling law for the average amplitude of the mean field

$$\langle R \rangle = N^{-1/4} F(\alpha N^{1/2}), \quad (11)$$

where $\alpha = \varepsilon - \varepsilon_c$ is the bifurcation parameter. We check this relation numerically in Fig. 1. One can see that the scaling relation (11) is valid in a wide range of the coupling constant ε.

We now discuss the correlations of the mean field. As Eq. (7) describes the noisy Hopf-Andronov bifurcation, one can directly apply here the results of the theory of noisy self-sustained oscillations [17]. To the best of our knowledge, a clear picture exists only for states far beyond the Hopf-Andronov bifurcation point. The reason is that only far beyond the transition point can one separate the dynamics of

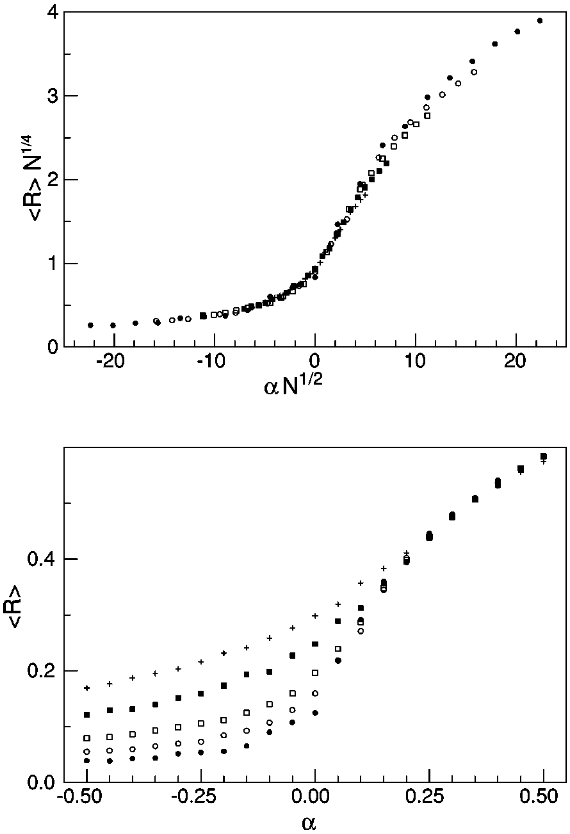

FIG. 1. Average amplitude of the mean field R in the original (bottom panel) and scaled (top panel) representations for $D=1$ and various population sizes: crests, $N=100$; filled squares, $N=200$; open squares, $N=500$; open circles, $N=1000$; and filled circles, $N=2000$.

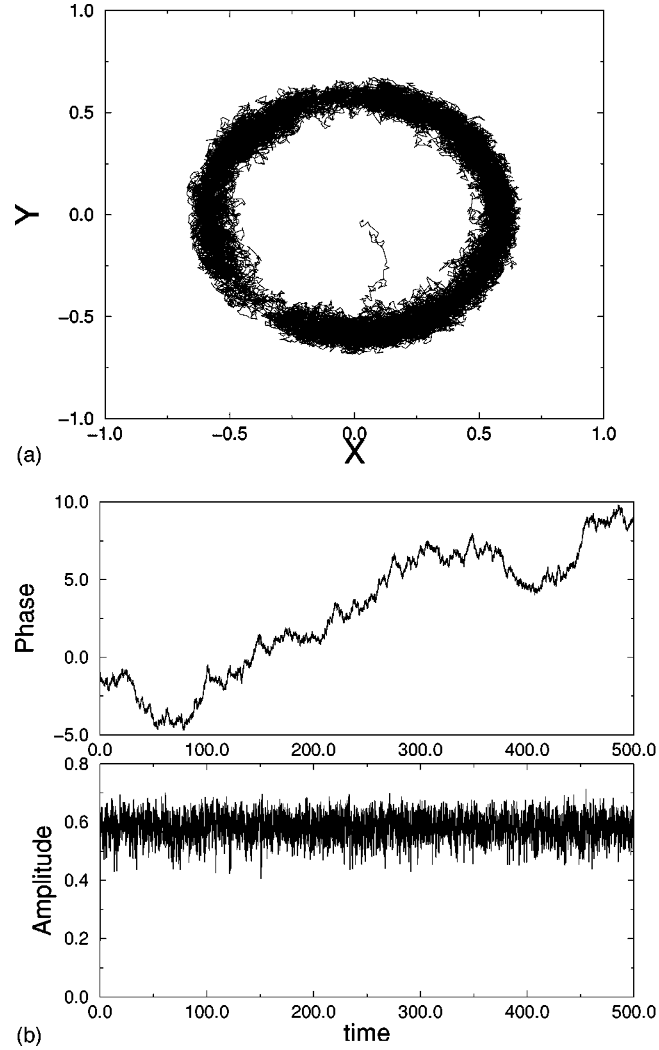

FIG. 2. Time evolution of the mean field for $\varepsilon=2.5$, $D=1$, and $N=500$. (a) The ''phase portrait'' on the X-Y plane: After initial transients the trajectory fills a ring of width proportional to $N^{-1/2}$. (b) The time dependence of the phase and the amplitude.

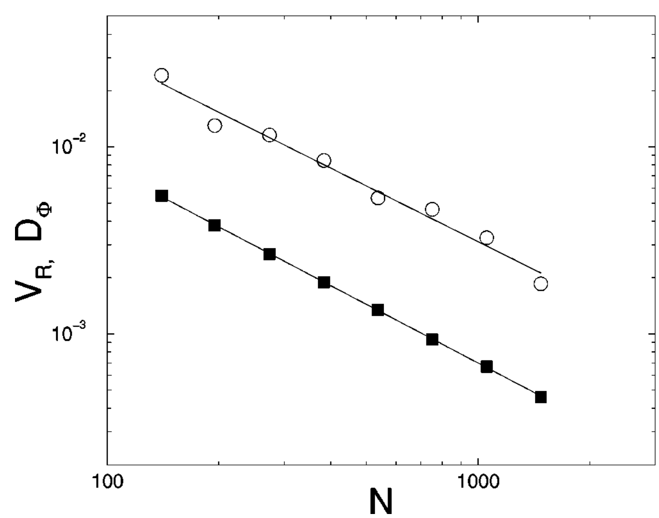

FIG. 3. Dependence of the variance of the amplitude V_R (squares) (the slope of the fit is -1.04) and the phase diffusion constant D_Φ (circles) (the slope of the fit is -0.99) on the population size N for $D=1$ and $\varepsilon=2.5$.

the phase from that of the amplitude. Indeed, for large α the probability distribution of the amplitude (10)

$$W_0(R) \propto R \exp\left[\frac{N}{\sigma}\left(\frac{\alpha}{4}R^2 - \frac{\varepsilon^2}{64D}R^4\right)\right]$$

has a sharp maximum at $R_{max} \approx 4\sqrt{D\alpha}\varepsilon^{-1}$. The phase of this state is well defined (because the amplitude does not vanish) and its distribution is uniform. On the plane Re(M)-Im(M) the trajectory of the mean field $M(t)$ fills a narrow (width proportioanl to $N^{-1/2}$) circle around R_{max} (see Fig. 2). Fluctuations of the amplitude are small and fast, while the phase exhibits a slow random-walk motion; see Fig. 2. As it has been shown in [17], both the variance of the amplitude and the diffusion constant of the phase are proportional to the intensity of the noise, i.e., in our case to N^{-1}. We confirm these scalings with numerical simulations in Fig. 3.

In conclusion, we have demonstrated that finite-size effects near the transition point in the population of globally coupled oscillators can be described using the noisy normal form equation. The effective noise in this equation scales with the system size as N^{-1}. In the vicinity of the bifurcation the fluctuations of the macroscopic mean field obey the scaling relation (11). Beyond the transition, one can separate the fluctuations of the amplitude and the phase; both scale as N^{-1}. The scaling predictions can be directly applied to the interpretation of possible experiments with physical [3], chemical, or biological [1,2] systems. For example, in laser arrays [4], the gain in the radiation quality depends on the number of coupled lasers. On the other hand, in some cases a comparison of the fluctuations of individual oscillators with those of the mean field could allow one to estimate the effective number of interacting subsystems.

The finite-size scaling above differs from the analogous results for the population of deterministic phase oscillators [9], where the effective fluctuations are different on the two sides of the transition. One possible reason for this discrepancy is that in the deterministic case one can clearly separate all oscillators into entrained and nonentrained, with different contributions to fluctuations. Another difference is in the very definition of the order parameter: In [9] it is not gauge invariant; in this work it is assumed that $\langle M\rangle \neq 0$ and this quantity is taken as an order parameter, while from the phase diffusion picture (Fig. 2) it follows $\langle M\rangle = 0$. It is suggestive to investigate the relation between the two problems in more detail. Also the question of finite-size effects in populations of oscillators with both noise and distribution of natural frequencies remains unsolved.

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